

REVISITING THE MODULI SPACE OF SEMISTABLE G -BUNDLES OVER AN ELLIPTIC CURVE

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ABSTRACT. This paper is concerned with the description of the moduli space of semistable G -bundles on an elliptic curve for a reductive group G . We show that it can be described in terms of line bundles on the elliptic curve and a certain relative Weyl group. This generalizes the method of Laszlo [Las98] and recovers the (global) description of the moduli space due to Friedman, Morgan, Witten [FM98, FMW98]. The proof is algebraic and works in arbitrary characteristic. Along the way we show that the main results of [Fra16] extend to positive characteristic.

1. INTRODUCTION

1.1. The study of principal bundles on elliptic curves began with the seminal paper of Atiyah [Ati57] where he gave a complete and beautiful description of all the semistable vector bundles. He didn't discuss the moduli space but one could have easily guessed from his results the precise statement. Let us denote by \mathcal{M}_r^d the moduli space of semistable vector bundles of rank r and degree d . In case r and d are coprime Atiyah essentially proved that the determinant map

$$\det : \mathcal{M}_r^d \rightarrow \mathcal{M}_1^d \quad (1)$$

is an isomorphism of algebraic varieties.

In general, if we denote by $m = \gcd(r, d)$, we have

$$\mathcal{M}_r^d \simeq E^m / \mathfrak{S}_m \quad (2)$$

where E is the elliptic curve and \mathfrak{S}_m is the symmetric group on m letters (the isomorphism is not canonical however). The isomorphisms (1) and (2) hold in any characteristic and the proof, in characteristic 0, appeared in a paper by Tu [Tu93, Theorem 1] where he also proves a Verlinde formula among other interesting results.

Laszlo [Las98] generalized the isomorphism (2) to the case of reductive groups and trivial connected component. Namely, let G be a reductive group and denote by \mathcal{M}_G^0 the moduli space of semistable G -bundles on E of degree 0 (i.e. topologically trivial, or zero first Chern class). Then we have

$$\mathcal{M}_G^0 \simeq (\mathbb{X}_*(T) \otimes_{\mathbb{Z}} \text{Pic}^0(E)) / W$$

where we have denoted by T a maximal torus of G , by $\mathbb{X}_*(T)$ the group of its cocharacters and by W the Weyl group. His proof is through a Birkhoff-Grothendieck type result which says that every semistable G -bundle of degree 0 over an elliptic curve is an extension of line bundles of degree 0. Looijenga has proved [Loo76] that the RHS above is a weighted projective space where the weights can be read off the combinatorics of the root system of G .

Concerning the other components of the moduli space, motivated by 2d conformal field theory, Schweigert has proved in [Sch96] that for any given topological type, say $c \in \pi_1(G)$, there is another reductive group, call it G_c , such that $\mathcal{M}_G^c \simeq \mathcal{M}_{G_c}^0$. His statements are in the realm of differential geometry but one could possibly find a more algebro geometric approach (this is a topic under investigation joint with Sam Gunningham and Penghui Li). Coupled with the above mentioned theorem of Laszlo one can therefore deduce a description of the moduli space of semistable G -bundles of non-zero degree in terms of line bundles and a certain Weyl group.

Another take on this problem has been given by Friedman, Morgan and Witten in a series of papers [FM98, FMW98, FM00]. They have two approaches: one is analytic through flat bundles which is very hands-on and adapted to concrete computations, however not very suitable to questions regarding moduli spaces (or universal bundles). In their second approach, which uses deformation theory and is more algebraic, they provided a description of \mathcal{M}_G^c as a weighted projective space, thus recovering also Looijenga's theorem. However, their method is very different from Laszlo's but one upshot is the construction, in some cases, of a universal bundle.

Our goal is to give a new proof for the global description of \mathcal{M}_G^c by generalising Laszlo's approach. Namely, we show that for a reductive group G and a topological type $c \in \pi_1(G)$, there is a Levi subgroup L_c such that each semistable G -bundle is S -equivalent¹ to the induction to G of a **stable** L_c -bundle. Then, following Laszlo's strategy, we conclude that

$$\mathcal{M}_G^c \simeq \mathcal{M}_{L_c}^{c'} / W_{L_c, G} \quad (3)$$

where $W_{L_c, G} = N_G(L_c) / L_c$ is the relative Weyl group. Moreover, generalising (1), we show that the determinant map

$$\det : \mathcal{M}_{L_c}^{c'} \rightarrow \mathcal{M}_{L_c/[L_c, L_c]}^{\det(c')} \quad (4)$$

is an isomorphism of algebraic varieties. The two isomorphism together provide a description of \mathcal{M}_G^c in terms of line bundles and a certain Weyl group.

Some of the advantages of this approach over those in [FM98, FMW98] are that it also works in positive characteristic and the proofs in this paper are uniform with respect to the Dynkin type of the group G . To obtain precise information on L_c above we do use however some results from [Fra16], namely Corollary 4.3 and Section 4.2 from loc. cit., that are done by inspecting the combinatorics of each root system. Another advantage is the close similarity with the more intuitive situation of vector bundles, compare isomorphism (1) vs (4) and (2) vs (3),

However, we do not address in this paper the existence/construction of universal bundles. Indeed, the isomorphism (3) doesn't seem adapted to answering this question since the universal bundle on $\mathcal{M}_{L_c}^{c'}$, if it exists, doesn't descend to \mathcal{M}_G^c . For a more thoroughful discussion of universal bundles we invite the reader to look at [FMW98]. See also Remarks 3.11, 4.6.

1.2. Before giving the main statement let us introduce some notation.

We'll be working over an algebraically closed field \mathbf{k} , E is a smooth projective curve of genus 1 over \mathbf{k} and G is a reductive group over \mathbf{k} . We fix a Borus $T \subset B \subset G$.

¹This is the classical equivalence relation on the set of semistable bundles needed in order to have a moduli space.

We denote by $\mathbb{X}_*(T)$ the group of cocharacters of T . Let us recall that for a parabolic subgroup $B \subset P \subset G$, the algebraic fundamental group $\pi_1(P)$ is given by $\mathbb{X}_*(T)/\langle \check{\alpha} \text{ coroot of } P \rangle$. We'll denote by $\check{\lambda}_P$ an element of $\pi_1(P)$.

We denote by $\text{Bun}_G^{\text{sst}}$ and by \mathcal{M}_G the moduli stack, respectively moduli space, of semistable G -bundles over E . Their connected components are labeled by elements of $\pi_1(G)$, see [Hof10]. We'll write $\text{Bun}_G^{\check{\lambda}_G, \text{sst}}$ and $\mathcal{M}_G^{\check{\lambda}_G}$ for such a connected component. Each such connected component is of finite type. In [BP03] it was proved under some restrictions on the characteristic of the field that $\mathcal{M}_G^{\check{\lambda}_G}$ exists as a normal projective variety. More generally, the existence and normality of the moduli space was proved in arbitrary characteristic in [GLSS08] (see Section 1.1 Main Theorem). For projectivity, in [GLSS08, Section 1.2] some assumptions on the characteristic of the field was needed. However, Heinloth showed in [Hei08, Hei10] that the projectivity holds over arbitrary fields.

We have a canonical map $\text{Bun}_G^{\check{\lambda}_G, \text{sst}} \rightarrow \mathcal{M}_G^{\check{\lambda}_G}$ which identifies two semistable G -bundles if their associated polystable G -bundles² is the same and kills all the automorphisms.

1.3. Here are the main results of this paper. We fix E an elliptic curve and G a reductive group, both over an algebraically closed field of arbitrary characteristic.

Theorem 1.1. *Let $\check{\lambda}_G \in \pi_1(G)$ be a fixed topological type. Then there exists a Levi subgroup $L = L_{\check{\lambda}_G} \subset G$ (unique up to conjugation) and $\check{\lambda}_L \in \pi_1(L)$ with the following properties:*

- (1) ([Fra16]) *all the semistable L -bundles in $\mathcal{M}_L^{\check{\lambda}_L}$ are stable, in particular the S -equivalence relation reduces to isomorphism classes.*
- (2) $\mathcal{M}_L^{\check{\lambda}_L} \rightarrow \mathcal{M}_G^{\check{\lambda}_G}$ *is a finite map, generically Galois, with Galois group the relative Weyl group $W_{L,G} = N_G(L)/L$.*
- (3) *the following natural map is an isomorphism*

$$\mathcal{M}_L^{\check{\lambda}_L} / W_{L,G} \simeq \mathcal{M}_G^{\check{\lambda}_G}.$$

Theorem 1.2. *Let L and $\check{\lambda}_L$ be as in the previous theorem. Then*

$$\det : \mathcal{M}_L^{\check{\lambda}_L} \rightarrow \mathcal{M}_{L/[L,L]}^{\det(\check{\lambda}_L)} \quad (5)$$

is an isomorphism.

Corollary 1.3. *Let $\check{\lambda}_G \in \pi_1(G)$ and $L, \check{\lambda}_L$ as in Theorem 1.1. Then we have*

$$\mathcal{M}_G^{\check{\lambda}_G} \simeq \mathcal{M}_{L/[L,L]}^{\det(\check{\lambda}_L)} / W_{L,G}. \quad (6)$$

Remark 1.4. For a torus Z we have $\mathcal{M}_Z^0 \simeq \text{Pic}^0(E) \otimes \mathbb{X}_*(Z)$ and we see therefore that $\mathcal{M}_G^{\check{\lambda}_G}$ can be described in terms of line bundles and a Weyl group.

In particular, this theorem recovers Laszlo's result since for $\check{\lambda}_G = 0$ the Levi L_0 is just the maximal torus. It also recovers the result of Tu because for $G = \text{GL}_n$ and $\check{\lambda}_G \equiv d$ we have that $L = (\text{GL}_{n/m})^m$ and $W_{L,G} = \mathfrak{S}_m$, where $m := \gcd(d, n)$. It is not possible to compare directly our description of $\mathcal{M}_G^{\check{\lambda}_G}$ with the one of Schweigert [Sch96] or Friedman, Morgan, Witten [FM98, FMW98] since there's no obvious

²For a semistable G -bundle \mathcal{F}_G the associated polystable G -bundle are the G -bundle $\text{gr}(\mathcal{F}_G)$ that is the unique closed point of $\{\overline{\mathcal{F}_G}\} \subset \text{Bun}_G^{\check{\lambda}_G, \text{sst}}$.

relationship between $\mathcal{M}_{L, \check{\lambda}_G}^{\check{\lambda}_L}$ and $\mathcal{M}_{G, \check{\lambda}_G}^0$ (see the beginning of the Introduction). However, one can check that the Weyl group of $G_{\check{\lambda}_G}$ is the same as our relative Weyl group $W_{L, G}$ and the maximal torus of $G_{\check{\lambda}_G}$ corresponds to the center $Z(L)$. A better understanding of the relationship is desirable.

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2. PRELIMINARIES.

2.1. Notations. For some of the notations see the last paragraph of the introduction. Here are a few more that we'll be using. By a G -bundle we mean a G -torsor in the fppf topology over the scheme/stack in question. Over a curve this is the same as étale G -torsors for G a smooth group. If \mathcal{F}_G is a G -bundle over B and F is a quasi-projective variety with a G action (e.g. a representation) then we denote by $F_{\mathcal{F}_G}$ the associated fiber space over B with fiber F . In particular, if V is a representation of G , we have the associated vector bundle $V_{\mathcal{F}_G}$.

We'll denote by X a smooth projective curve over \mathbf{k} . When we say curve we always mean a smooth projective curve over \mathbf{k} . Some results and definitions make sense for any genus so we'll state them like that.

For an algebraic group H we denote by $\mathbb{B}(H) = \mathbf{pt}/H$ the classifying stack of H -bundles. We denote by $\mathrm{Bun}_G(X)$ the moduli stack of G -bundles on X and by $\mathcal{M}_G(X)$ the corresponding moduli space (existence in arbitrary characteristic is proved in [GLSS08]). Similarly for the other groups T, B, P , etc. When we omit X and write Bun_G or \mathcal{M}_G we mean $\mathrm{Bun}_G(E)$ or $\mathcal{M}_G(E)$ where E is an elliptic curve. The connected components of $\mathrm{Bun}_G(X)$ are labelled by $\pi_1(G)$ (see [Hof10]).

Let us begin by giving some definitions and citing some results that we'll be using throughout the paper.

2.2. The slope map.

Definition 2.1. [see [Sch14]] For a parabolic subgroup $B \subset P \subset G$ with Levi subgroup L we define the slope map $\phi_P : \pi_1(P) \rightarrow \mathbb{X}_*(T)_{\mathbb{Q}}$ as follows

$$\pi_1(P) \rightarrow \pi_1(P)_{\mathbb{Q}} \simeq \mathbb{X}_*(Z(L))_{\mathbb{Q}} \rightarrow \mathbb{X}_*(T)_{\mathbb{Q}}$$

where we indicated by a subscript \mathbb{Q} the tensoring $\otimes_{\mathbb{Z}} \mathbb{Q}$.

For example, if $G = \mathrm{GL}_n$ and $\check{\lambda}_i, i = 1, \dots, n$ are the coordinate cocharacters of the diagonal matrices then $\pi_1(G) \simeq \mathbb{Z}\check{\lambda}_1$ and $\phi_G(d\check{\lambda}_1) = \frac{d}{n}(\check{\lambda}_1 + \dots + \check{\lambda}_n)$.

The slope map has some very nice properties and we refer the interested reader to [Sch14].

2.3. Semistability.

Definition 2.2. Let $H \subset K$ be a pair of algebraic groups and let $\mathcal{F}_K \rightarrow Y$ be a K -bundle over Y . A reduction of \mathcal{F}_K to H is a couple (\mathcal{F}_H, θ) of an H -bundle and an isomorphism $\theta : \mathcal{F}_H \overset{H}{\times} K \simeq \mathcal{F}_K$. Two reductions $(\mathcal{F}_H, \theta), (\mathcal{F}'_H, \theta')$ are equivalent if there is an isomorphism of H -bundles $\mathcal{F}_H \rightarrow \mathcal{F}'_H$ such that its extension to K intertwines θ and θ' .

Remark 2.3. To give a reduction of a K -bundle \mathcal{F}_K to H is the same as to give a section of $\mathcal{F}_K/H \rightarrow Y$. Two such sections give equivalent reductions if and only if there exists an automorphism $\sigma \in \text{Aut}(\mathcal{F}_K)$ translating one into the other.

Remark 2.4. For example, if $K = \text{GL}_n$ and H is the subgroup of uppertriangular matrices then to give a reduction to H of a rank n vector bundle (i.e. a GL_n -bundle) is the same as to give a filtration with subquotients being line bundles.

The following definition of semistability for G -bundles is from [Sch14] where it is also proved the equivalence with the Ramanathan's semistability.

Definition 2.5. G -bundle \mathcal{F}_G of degree $\check{\lambda}_G$ over a smooth projective curve X is (semi)stable if for any proper parabolic subgroup $P \subset G$ and for any reduction \mathcal{F}_P of \mathcal{F}_G to P of degree $\check{\lambda}_P$ we have

$$\phi_P(\check{\lambda}_P) \underset{(\leq)}{<} \phi_G(\check{\lambda}_G).$$

Proposition 2.6. [Sch14, Proposition 3.2 (b)] *If V is a highest weight representation of G of highest weight λ and \mathcal{F}_G is a G -bundle of degree $\check{\lambda}_G$ over a curve X then the slope (i.e degree divided by rank) of the associated vector bundle $V_{\mathcal{F}_G}$ is $\langle \phi_G(\check{\lambda}_G), \lambda \rangle$.*

2.4. Frobenius semistability. In case $\text{char}(\mathbf{k}) = p$ we will need moreover the notion of Frobenius semistable. Denote by $F_X : X \rightarrow X$ the absolute Frobenius: it's the identity at the level of topological spaces and raising to the power p at the level of functions.

Definition 2.7. A G -bundle \mathcal{F}_G is Frobenius semistable if $(F^n)^*(\mathcal{F}_G)$ is semistable for all $n \geq 0$.

Proposition 2.8. [Sun99, Proposition 1.2] *A G -bundle \mathcal{F}_G on a curve X is Frobenius semistable if and only if for any parabolic subgroup scheme (i.e. non necessarily reduced) P and any reduction of \mathcal{F}_G to P of degree $\check{\lambda}_P$ we have*

$$\phi_P(\check{\lambda}_P) \underset{(\leq)}{<} \phi_G(\check{\lambda}_G).$$

Remark 2.9. X. Sun proved also, see [Sun99, Theorem 2.1], that for genus 1 curves every semistable G -bundle is also Frobenius semistable.

2.5. Jordan-Hölder series. In the case of vector bundles it makes sense to talk about the category of semistable vector bundles of fixed slope. This is a finite length category so we can also talk about Jordan-Hölder series. To give a filtration of a vector bundle is the same as to give a reduction of the corresponding GL_n -bundle to a certain parabolic subgroup. In general the Jordan-Hölder series has no reason to have the same ranks and degrees of the graded parts when the vector bundle varies. However, this is a particularity of elliptic curves. Namely, it can be extracted from Atiyah's paper [Ati57] that for semistable vector bundles of rank n and degree d there is a (unique up to conjugation) parabolic such that all the semistable vector bundles of rank n and degree d admit a reduction to it and moreover the graded parts are stable vector bundles of equal slope. For example, for slope 0, all semistable vector bundles are extensions of degree 0 line bundles.

The following is an analogue for any reductive group G and any degree $\check{\lambda}_G$.

Theorem 2.10. *[Fra16, Lemma 2.12, Theorem 3.2, Corollary 4.2] Let $\check{\lambda}_G \in \pi_1(G)$ and consider $\text{Bun}_G^{\check{\lambda}_G, \text{sst}}$ the stack of semistable G -bundles of degree $\check{\lambda}_G$ on an elliptic curve E . Then there exists a unique (up to conjugation) parabolic subgroup P and a unique $\check{\lambda}_P \in \pi_1(P)$ such that*

- (1) $\phi_G(\check{\lambda}_G) = \phi_P(\check{\lambda}_P)$,
- (2) every semistable G -bundle of degree $\check{\lambda}_G$ has a reduction to P of degree $\check{\lambda}_P$,
- (3) the map

$$\text{Bun}_P^{\check{\lambda}_P, \text{sst}} \rightarrow \text{Bun}_G^{\check{\lambda}_G, \text{sst}}$$

is proper, generically Galois with Galois group $W_{L,G} = N_G(L)/L$ where L is the Levi subgroup of G .

- (4) for any $\mathcal{F}_P \in \text{Bun}_P^{\check{\lambda}_P, \text{sst}}$ the induced L -bundle is stable.
- (5) ([Fra16, Corollary 4.3]) For a reductive group L and $\check{\lambda}_L \in \pi_1(L)$ there exist stable L -bundles of degree $\check{\lambda}_L$ if and only if $L^{\text{ad}} = \prod_i \text{PGL}_{n_i}$ and $\check{\lambda}_L^{\text{ad}} \equiv (d_i)_i$ with $\gcd(d_i, n_i) = 1, \forall i$.

Remark 2.11. In [Fra16] there is a table with all the possible subgroups L that appear in the above theorem. For the convenience of the reader we make a copy of the table in the Appendix.

Remark 2.12. The proof from [Fra16] is in characteristic zero, however the only moment that we used it was to apply "generic smoothness" (see [Fra16, Lemma 3.9]) and deduce the existence of certain regular bundles (see Definition 3.6) which we prove here in arbitrary characteristic (see Lemma 3.9). Therefore the results of [Fra16] hold in positive characteristic as well.

2.6. Vector bundles over an elliptic curve.

Theorem 2.13. *[Ati57, Corollary to Theorem 7] Let $n \geq 1$ and $d \in \mathbb{Z}$ be coprime. Then we have*

- (1) Any stable rank n degree d vector bundle over E is uniquely determined by its determinant bundle.
- (2) If \mathcal{V} is a vector bundle as above and $\mathcal{L} \in \text{Pic}^0(E)$ then $\mathcal{V} \otimes \mathcal{L} \simeq \mathcal{V}$ if and only if $\mathcal{L} \in \text{Pic}^0(E)[n]$.

Theorem 2.14. *[BH10, Lemma 2.2.1 and Example 5.1.4] Let X be a smooth projective curve and let*

$$1 \rightarrow Z \rightarrow G \rightarrow H \rightarrow 1$$

be a central extension. Fix $\check{\lambda}_G \in \pi_1(G)$ and denote by $\check{\lambda}_H$ the image of $\check{\lambda}_G$ in $\pi_1(H)$. Then the map

$$\text{Bun}_G^{\check{\lambda}_G}(X) \rightarrow \text{Bun}_H^{\check{\lambda}_H}(X)$$

is a $\text{Bun}_Z^0(X)$ -torsor. The same holds for semistable bundles also since being semistable for G or H is the same thing (the flag varieties are the same).

Corollary 2.15. *Let $n \geq 1$ and $d \in \mathbb{Z}$ be coprime. Then over an elliptic curve E we have*

$$\text{Bun}_{\text{PGL}_n}^{d, \text{st}} \simeq \mathbb{B}(\text{Pic}^0(E)[n]),$$

where we have denoted by $\text{Pic}^0(E)[n]$ the kernel (subgroup scheme) of the multiplication by $n : \text{Pic}^0(E) \rightarrow \text{Pic}^0(E)$. In particular, we deduce that $\mathcal{M}_{\text{PGL}_n}^d = \text{pt}$.

Proof. Using Theorem 2.14 we have that $\text{Bun}_{\text{GL}_n}^{d,\text{st}} \rightarrow \text{Bun}_{\text{PGL}_n}^{d,\text{st}}$ is a $\text{Bun}_{\mathbb{G}_m}^0$ -torsor. Using also Theorem 2.13 we deduce that $\text{Bun}_{\text{PGL}_n}^{d,\text{st}}$ has only one isomorphism class of objects and the automorphism group is the kernel of the action of $\text{Pic}^0(E)$ on $\text{Bun}_{\text{GL}_n}^{d,\text{st}}$ which again by Theorem 2.13 is exactly $\text{Pic}^0(E)[n]$. \square

3. PROOF OF THEOREM 1.1.

3.1. The action of the center.

Corollary 3.1. *Let E be an elliptic curve, let L be a reductive group and let $\check{\lambda}_L \in \pi_1(L)$ such that there exist stable L -bundles of degree $\check{\lambda}_L$ on E (see Theorem 2.10 (5)). Then the action of $\mathcal{M}_{Z(L)}^0$ on $\mathcal{M}_L^{\check{\lambda}_L}$ is transitive.*

Proof. From Theorem 2.10 (5) we have that $L^{\text{ad}} \simeq \prod_i \text{PGL}_{n_i}$ and $\check{\lambda}_L^{\text{ad}} = (d_i)_i$ such that $\gcd(d_i, n_i) = 1$. So we can apply Corollary 2.15 to conclude that $\mathcal{M}_{L^{\text{ad}}}^{\check{\lambda}_L^{\text{ad}}} = \text{pt.}$

Since $\text{Bun}_L^{\check{\lambda}_L, \text{st}} \rightarrow \text{Bun}_{L^{\text{ad}}}^{\check{\lambda}_L^{\text{ad}}, \text{st}}$ is a $\text{Bun}_{Z(L)}^0$ -torsor (see Theorem 2.14) we deduce that $\text{Bun}_{Z(L)}^0$ acts on $\text{Bun}_L^{\check{\lambda}_L, \text{st}}$ transitively on objects. This property is clearly preserved when we pass to moduli spaces. \square

Corollary 3.2. *Under the same hypotheses as the previous corollary we have that all $\mathcal{F}_L \in \text{Bun}_L^{\check{\lambda}_L, \text{st}}$ have the same automorphism group.*

Proof. We put $Z := Z(L)$. For $\mathcal{F}_Z \in \text{Bun}_Z^0$ and $\mathcal{F}_L \in \text{Bun}_L$ we have a canonical isomorphism $\text{Aut}(\mathcal{F}_L) \rightarrow \text{Aut}(\mathcal{F}_L \otimes \mathcal{F}_Z)$ sending θ to $\theta \otimes \text{id}$. From Corollary 3.1 we have that $\text{Bun}_Z^0 \curvearrowright \text{Bun}_L^{\check{\lambda}_L, \text{st}}$ transitively on objects so we conclude. \square

Remark 3.3. The above Corollary is never used in the sequel but it allows us to see that $\text{Bun}_L^{\check{\lambda}_L, \text{st}} \rightarrow \mathcal{M}_L^{\check{\lambda}_L}$ is a gerbe.

3.2. Regular bundles. This subsection is dedicated to proving the following result:

Lemma 3.4. *Let $\check{\lambda}_G \in \pi_1(G)$ and $L, P, \check{\lambda}_L \in \pi_1(L)$ as in Theorem 2.10. Then the map $\mathcal{M}_L^{\check{\lambda}_L} \rightarrow \mathcal{M}_G^{\check{\lambda}_G}$ is generically étale.*

Remark 3.5. By looking at the tangent spaces one needs to prove the existence of an L -bundle of degree $\check{\lambda}_L$ such that $H^*(E, (\mathfrak{g}/\mathfrak{l})_{\mathcal{F}_L}) = 0$.

Let us introduce the notion of regular L -bundles.³

Definition 3.6. Let H be an algebraic group and V a representation of H . Consider $\check{\lambda}_H \in \pi_1(H)$ such that its image in $\pi_1(\text{GL}(V))$ is 0. An H -bundle \mathcal{F}_H of degree $\check{\lambda}_H$ is called V -regular if $H^0(X, V_{\mathcal{F}_H}) = 0$.

Let $P \subset G$ be a parabolic subgroup with Levi subgroup L . A P -bundle \mathcal{F}_P of degree $\check{\lambda}_P$ over a curve X is called regular if it is $\mathfrak{g}/\mathfrak{p}$ -regular. An L -bundle is called regular if it is $\mathfrak{g}/\mathfrak{l}$ -regular. When there's no confusion we'll just say regular.

Remark 3.7. This condition ensures that the differential of $\text{Bun}_P^{\check{\lambda}_P} \rightarrow \text{Bun}_G^{\check{\lambda}_G}$ is injective at this point, or in other words that the map is an immersion at this point.

³There exists another notion of regular stable bundles: those whose automorphism group is exactly the center of the group (see [FM98, FMW98]). However we'll not use this notion in this paper.

We'll use the following result which holds over any curve X :

Lemma 3.8. *[Sun99, Corollary 1.1] Let \mathcal{F}_G be a Frobenius semistable G -bundle over a smooth projective curve X and let $f : G \rightarrow G'$ be a morphism of reductive groups such that $f(Z(G)) \subset Z(G')$. Then the induced G' -bundle is also Frobenius semistable. In particular, if V is a representation of G such that the center of G acts by a character, the induced vector bundle $V_{\mathcal{F}_G}$ is semistable.*

The core of the proof of Lemma 3.4 is to show that there exist regular bundles:

Lemma 3.9. *Let X be a curve and $\check{\lambda}_G, P, L, \check{\lambda}_P$ as in Theorem 2.10. Then the substack of regular L -bundles in $\text{Bun}_L^{\check{\lambda}_L, \text{st}}$ is open and dense.*

Proof. The strategy is the following: we start with an arbitrary Frobenius semistable L -bundle (see Remark 2.9 for existence) and we tensor it with a sufficiently generic $Z := Z(L)$ -bundle of degree 0 to produce a regular L -bundle.

The openness follows from the semicontinuity of $\dim(H^0(X, (\mathfrak{g}/\mathfrak{l})_{\mathcal{F}_L}))$ so all we need to prove is the nonemptiness of the regular locus.

More precisely, let \mathcal{F}_L be a Frobenius stable L -bundle of degree $\check{\lambda}_L$ and V be a highest weight representation of L such that $V_{\mathcal{F}_L}$ is of degree 0 and such that the center $Z = Z(L)$ acts on V by a nontrivial character χ . Lemma 3.8 guarantess that $V_{\mathcal{F}_L}$ is semistable of degree 0 and hence the set of line subbundles of degree 0 of $V_{\mathcal{F}_L}$ is finite.

Now let us consider a Z -bundle \mathcal{F}_Z of degree 0. Using the group morphism $Z \times L \rightarrow L$ we can produce a new L -bundle that we denote $\mathcal{F}_L \otimes \mathcal{F}_Z$ which is still Frobenius semistable of degree $\check{\lambda}_G$. Since Z acts on V by χ we have that $V_{\mathcal{F}_L \otimes \mathcal{F}_Z} = V_{\mathcal{F}_L} \otimes \chi_{\mathcal{F}_Z}$ and so the set of line subbundles of degree 0 of $V_{\mathcal{F}_L \otimes \mathcal{F}_Z}$ is the one for $V_{\mathcal{F}_L}$ tensored by $\chi_{\mathcal{F}_Z}$. Since χ is non-trivial we obtain that for almost all Z -bundles the trivial line bundle \mathcal{O} is not a line subbundle of $V_{\mathcal{F}_L \otimes \mathcal{F}_Z}$, in other words $H^0(X, V_{\mathcal{F}_L \otimes \mathcal{F}_Z}) = 0$. So we've produced an open dense substack of L -bundles \mathcal{F}_L of degree $\check{\lambda}_P$ that are V -regular.

Now we will apply this to the representation $L \curvearrowright \mathfrak{g}/\mathfrak{l}$. It is not a highest weight representation but it admits a filtration with subquotients of highest weight. Since the weights of $\mathfrak{g}/\mathfrak{l}$ are among the roots of \mathfrak{g} we have that if W is such a subquotient then $W_{\mathcal{F}_L}$ is semistable of degree 0 provided that \mathcal{F}_L is Frobenius stable (see Lemma 3.8 and Proposition 2.6). Also the central characters are not trivial because the centraliser of $Z(L)$ in G is precisely L . Therefore, by the previous paragraph, for each such subquotient W the substack of W -regular L -bundles is open and dense.

Intersecting all of these open dense substaks we have that the regular L -bundles form an open dense substack in $\text{Bun}_L^{\check{\lambda}_L, \text{sst}}$. \square

Proof. (of Lemma 3.4) Proving generic étaleness is equivalent to proving the map is étale at some point, say \mathcal{F}_L . By looking at the differential of the map we have to show the bijectivity of

$$H^1(E, \mathfrak{l}_{\mathcal{F}_L}) \rightarrow H^1(E, \mathfrak{g}_{\mathcal{F}_L}).$$

This is implied by the vanishing of $H^i(E, (\mathfrak{g}/\mathfrak{l})_{\mathcal{F}_L})$, $i = 0, 1$.

Let \mathcal{F}_L be a regular L -bundle (see Lemma 3.9). Then by definition we have $H^0(E, (\mathfrak{g}/\mathfrak{l})_{\mathcal{F}_L}) = 0$. By Riemann-Roch we get that $H^1(E, (\mathfrak{g}/\mathfrak{l})_{\mathcal{F}_L}) = \deg((\mathfrak{g}/\mathfrak{l})_{\mathcal{F}_L}) = 0$ where for the last equality we used genus 1 and Proposition 2.6. \square

Lemma 3.10. *Let $\check{\lambda}_G \in \pi_1(G)$ and $L, P, \check{\lambda}_L \in \pi_1(L)$ as in Theorem 2.10 and put $W := W_{L,G}$ the relative Weyl group of $L \subset G$. Then the map $\pi : \mathcal{M}_L^{\check{\lambda}_L} \rightarrow \mathcal{M}_G^{\check{\lambda}_G}$ is W -invariant and the fibers are W -orbits. In particular it is a finite map.*

Proof. Both moduli spaces are projective varieties so finiteness follows from quasi-finiteness which in turn follows from the fact that the fibers are W -orbits.

Remark that the map π is clearly W -invariant. Indeed, this is a general fact: an H -bundle doesn't change its isomorphism class when acted upon by an inner automorphism of H . In our case, the action of an element $w \in W = N_G(L)/L$ on L becomes an inner automorphism of G , so the isomorphism class of the induced G -bundle is not affected.

Let us prove now that the fibers are W -orbits. Let $\mathcal{F}_L, \mathcal{F}'_L \in \mathcal{M}_L^{\check{\lambda}_L}$ be two L -bundles in the fiber of π , namely $\mathcal{F}_L \times^L G \simeq \mathcal{F}'_L \times^L G$. Let us call \mathcal{F}_P and \mathcal{F}'_P the induced P -bundles.

Let us recall the notion of relative position: the bundles \mathcal{F}_P and \mathcal{F}'_P are (generically) in relative position \tilde{w} if the section $s : X \rightarrow \mathcal{F}_P \times^P G/P$ that gives \mathcal{F}'_P lands (generically) in $\mathcal{F}_P \times^P P\tilde{w}P/P$. We denote by \tilde{w} a representative of a coset in the double coset space $P \backslash G/P$.

Let us denote by \tilde{w} the generic relative position of $\mathcal{F}_P, \mathcal{F}'_P$ which we recall are of degree $\check{\lambda}_P$ (see Lemma 2.10). Lemma 3.5 from [Fra16] tells us that the two P -bundles are in relative position \tilde{w} and then Lemma 3.7 from loc.cit gives us moreover that $\tilde{w} \in N_G(L)/L = W_{L,G}$.

Since we have not only P -bundles but actually L -bundles we can conjugate one of them by \tilde{w} and therefore we can assume that \mathcal{F}_P and \mathcal{F}'_P are in general position 1.

So in order to finish the proof we need to show that $\mathcal{F}_L \simeq \mathcal{F}'_L$ provided the two induced P -bundles are in relative position 1. But being in relative position 1 means that the section giving \mathcal{F}'_P satisfies $s : X \rightarrow \mathcal{F}_P \times^P P/P = X$ and therefore $\mathcal{F}'_P \simeq \mathcal{F}_P$. By quotienting out by $U = R_u(P)$ we get $\mathcal{F}_L \simeq \mathcal{F}_P/U \simeq \mathcal{F}'_P/U \simeq \mathcal{F}'_L$ which is what we wanted. \square

3.3. Proof of Theorem 1.1.

Proof. We finish the proof of Theorem 1.1. The point (1) is contained in Theorem 2.10 (4).

To prove (2) we combine Theorem 2.10 (3), Lemma 3.4 and Lemma 3.10.

To prove (3), from Lemma 3.10 we have that the natural map $\mathcal{M}_L^{\check{\lambda}_L} \rightarrow \mathcal{M}_G^{\check{\lambda}_G}$ factorises through $\mathcal{M}_L^{\check{\lambda}_L}/W_{L,G}$ and moreover the morphism $\mathcal{M}_L^{\check{\lambda}_L}/W_{L,G} \rightarrow \mathcal{M}_G^{\check{\lambda}_G}$ is bijective and separable, see Lemma 3.4. Since the target is a normal variety (see [GLSS08]), we can apply Zariski's main theorem to conclude that it is an isomorphism. \square

Remark 3.11. One might think that if $\mathcal{M}_L^{\check{\lambda}_L}$ has a universal bundle then it descends to $\mathcal{M}_G^{\check{\lambda}_G}$. However, unless $L = G$, this is not the case and one reason is that the dimension of the automorphism group of a G -bundle induced from L varies (the jumps arise at non regular L -bundles).

4. PROOF OF THEOREM 1.2

Unless otherwise stated, in this section L is a reductive group and $\check{\lambda}_L \in \pi_1(L)$ such that $\text{Bun}_L^{\check{\lambda}_L, \text{st}}(E) \neq \emptyset$ where E is an elliptic curve.

The idea of the proof is to look at the action of $\mathcal{M}_{Z(L)}^0$ on $\mathcal{M}_L^{\check{\lambda}_L}$ and on $\mathcal{M}_{L/[L,L]}^{\det(\check{\lambda}_L)}$ and observe that in both cases it is transitive with the same stabilizer.

4.1. Preliminaries. Recall from Theorem 2.10 (5) that the assumption on L forces $L^{\text{ad}} \simeq \prod_i \text{PGL}_{n_i}$ for some n_i . We denote by $\text{cZ} = L/[L, L]$ the co-center of L and by $Z = Z(L)$ the center of L .

Let us recall that the natural map $\det : L \rightarrow \text{cZ}$ is called the determinant. The homomorphisms $Z \times L \rightarrow L$ and $Z \times \text{cZ} \rightarrow \text{cZ}$ naturally give an action of Bun_Z^0 on $\text{Bun}_L^{\check{\lambda}_L, \text{st}}$ and on $\text{Bun}_{\text{cZ}}^{\det(\check{\lambda}_L)}$.

For a diagonalizable group D we write $\text{Bun}_D^0(X)$ for the moduli stack of D -bundles \mathcal{F}_D on X such that for any character $\chi : D \rightarrow \mathbb{G}_m$ the associated line bundle $\chi_{\mathcal{F}_D}$ is of degree 0. If D is not a torus then this is not connected, for e.g. if $D = \mu_n$ then $\text{Bun}_{\mu_n}^0(X) = \text{Pic}^0(X)[n]/\mu_n$ the n -torsion in $\text{Pic}^0(X)$.

Also, for a diagonalizable group D , we denote by \mathcal{M}_D^0 the moduli space of D -bundles of degree 0 in the same sense as above. It is a group scheme whose (reduced) connected component of the identity is an abelian variety. For example, if D is a torus, then $\mathcal{M}_D^0 \simeq \text{Pic}^0(X)^{\dim(D)}$. If D has some finite component then \mathcal{M}_D^0 is a product of an abelian variety and a finite group scheme which is a finite subgroup of an abelian variety. For example, for $D = \mu_n$ we have that $\mathcal{M}_D^0 = \ker(n : \text{Pic}^0(X) \rightarrow \text{Pic}^0(X))$. Remark that in positive characteristic \mathcal{M}_D^0 might not be smooth.

4.2. Diagonalizable groups. Let us recall that a diagonalizable group is a group that is isomorphic to a product of several \mathbb{G}_m and μ_n for various $n \geq 2$. The category of diagonalizable groups is equivalent to the category of finitely generated abelian groups, the equivalence being given by taking the characters the diagonalizable group.

We'll need the following technical lemmas:

Lemma 4.1. *Let L be a reductive group such that $L^{\text{ad}} \simeq \prod_i \text{PGL}_{n_i}$. Assume that $[L, L]$ is simply connected. Then there exists a torus T' such that $L \hookrightarrow \prod_i \text{GL}_{n_i} \times T'$.*

Proof. The proof is essentially linear algebra.

We have $L = \prod_i \text{SL}_{n_i} \overset{C}{\times} Z(L)$, where $C = \prod_i \mu_{n_i}$.

There is a torus T' and a map $\phi : Z(L) \hookrightarrow \prod_i \mathbb{G}_m \times T'$ such that the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\quad} & Z(L) \\ \text{can} \downarrow & \searrow \phi & \\ \prod_i \mathbb{G}_m \times T' & & \end{array}$$

Indeed, using the equivalence of diagonalizable groups with finitely generated abelian groups we need to show that it exists $\phi : \prod_i \mathbb{Z} \times \mathbb{Z}^r \rightarrow M$ such that the

following diagram commutes

$$\begin{array}{ccc} \prod_i \mathbb{Z}/n_i & \xleftarrow{u} & M \\ \uparrow \text{can} & \nearrow \phi & \\ \prod_i \mathbb{Z} \times \mathbb{Z}^r & & \end{array}$$

where M is the abelian group corresponding to $Z(L)$.

This can be done easily as follows: first take $r = 0$ and use that $\prod_i \mathbb{Z}$ is free and u is surjective. Then, for a convenient $r \geq 0$ add \mathbb{Z}^r mapping surjectively onto $\ker(u)$. \square

Lemma 4.2. *Let L be an arbitrary reductive group. Then there exists a central extension*

$$1 \rightarrow T' \rightarrow \hat{L} \rightarrow L \rightarrow 1$$

with $[\hat{L}, \hat{L}]$ simply connected and T' a torus. In particular, since T' is connected, we have $\pi_1(\hat{L}) \twoheadrightarrow \pi_1(L)$.

Proof. We write $L = [L, L] \times^C Z = [L, L]^{\text{sc}} \times^{\tilde{C}} Z$ where $\tilde{C} = Z([L, L]^{\text{sc}})$ and $[L, L]^{\text{sc}}$ is the simply connected cover of $[L, L]$. Let us choose a torus T' and an inclusion $\tilde{C} \hookrightarrow T'$. We define the following group

$$\hat{L} := [L, L]^{\text{sc}} \times^{\tilde{C}} (Z \times T')$$

where $\tilde{C} \rightarrow Z \times T'$ is the diagonal homomorphism (injective!). Clearly

The natural homomorphism $\hat{L} \rightarrow L$, forgetting the factor T' , is surjective and its kernel is exactly T' . \square

4.3. The action of the center and proof of Theorem 1.2. In this subsection we'll analyse in detail the stabiliser of $\mathcal{M}_{Z(L)}^0$ acting on $\mathcal{M}_L^{\check{\lambda}_L}$ for $L, \check{\lambda}_L$ such that there exist stable bundles.

We start with a basic general lemma:

Lemma 4.3. *Let $H \subset L$ be reductive groups such that $[H, H] = [L, L]$. Let $\mathcal{F}_H, \mathcal{F}'_H$ be two H -bundles on a proper scheme Y . Then if the induced L -bundles are isomorphic the H -bundles are also.*

Proof. We'll see \mathcal{F}'_H as a reduction to H of the L -bundle $\mathcal{F}_H \times^H L$ (see Definition 2.2 and the remark following). So we have a section $s : Y \rightarrow \mathcal{F}_H \times^H L/H$. We need to show that by an automorphism of the L -bundle $\mathcal{F}_H \times^H L$ we can translate it into the trivial section $s_0 : Y = \mathcal{F}_H \times^H H/H \hookrightarrow \mathcal{F}_H \times^H L$.

From the assumptions we have that $HZ(L) = L$ hence H acts trivially on L/H . Therefore the section s can be seen as a section $s : Y \rightarrow Y \times L/H$, i.e. as a map $Y \rightarrow L/H$. As H, L are reductive the quotient L/H is an affine variety so s must be constant, say equal to \bar{z} , because Y is proper.

Now from the assumptions again $Z(L) \twoheadrightarrow L/H$ so we can take $z \in Z(L)$ a lift of \bar{z} . The element z being in the center of L gives an automorphism, call it θ_z of $\mathcal{F}_H \times^H L$ and from the choices we've made we have that $\theta_z^{-1}(s) = s_0$, or in other words that the section s gives an H -bundles isomorphic to \mathcal{F}_H . \square

Remark 4.4. The above Lemma is false if Y is not proper (think of modules of a Dedekind ring) and it is also false if L/H is not affine (e.g. two filtrations of the same vector bundle need not be isomorphic).

Lemma 4.5. *Let $L, \check{\lambda}_L$ be such that there exist stable bundles over an elliptic curve E (see Theorem 2.10 (5)). Then the stabiliser of $\mathcal{M}_{Z(L)}^0$ acting on $\mathcal{M}_L^{\check{\lambda}_L}$ is $\mathcal{M}_{Z([L, L])}$.*

Proof. First let us treat the case $[L, L]$ simply connected. Put $Z := Z(L)$. In this case, Lemma 4.1 allows us to see $L \subset \prod_i \mathrm{GL}_{n_i} \times T' := H$ where T' is a torus.

Let $\mathcal{F}_Z \in \mathcal{M}_Z^0$ and $\mathcal{F}_L \in \mathcal{M}_L^{\check{\lambda}_L}$ be such that $\mathcal{F}_Z \otimes \mathcal{F}_L \simeq \mathcal{F}_L$. We want to show that \mathcal{F}_Z has a reduction to $Z([L, L])$.

Using Lemma 4.3 we can see that \mathcal{F}_L being isomorphic to $\mathcal{F}_Z \otimes \mathcal{F}_L$ is equivalent to their induction to H being isomorphic, i.e. that $\mathcal{F}_Z \times^Z Z(H) \otimes \mathcal{F}_H \simeq \mathcal{F}_H$, where we put $\mathcal{F}_H := \mathcal{F}_L \times^L H$.

Since $[L, L] = [H, H]$ the bundle \mathcal{F}_H is a stable H -bundle so using Theorem 2.13 we have that $\mathcal{F}_{Z(H)}$ has a reduction to $Z([H, H])$. Another application of Lemma 4.3 gives that \mathcal{F}_Z has a reduction to $Z([H, H]) = Z([L, L])$ which is what we wanted to show.

Let us now deal with the general case. From Lemma 4.2 there is a central extension $1 \rightarrow T' \rightarrow \hat{L} \rightarrow L$ with T' a torus and $[\hat{L}, \hat{L}]$ simply connected. We put $\hat{Z} = Z(\hat{L})$.

Let $\check{\lambda}_{\hat{L}} \in \pi_1(\hat{L})$ be a lift of $\check{\lambda}_L$. From Theorem 2.14 we have that

$$\mathrm{Bun}_{\hat{L}}^{\check{\lambda}_{\hat{L}}, \mathrm{st}} \rightarrow \mathrm{Bun}_L^{\check{\lambda}_L, \mathrm{st}}$$

is a $\mathrm{Bun}_{T'}^0$ -torsor. In particular, if $\mathcal{F}_Z \otimes \mathcal{F}_L \simeq \mathcal{F}_L$ then, for some lifts, we'll have $\mathcal{F}_{\hat{Z}} \otimes \mathcal{F}_{\hat{L}} \simeq \mathcal{F}_{\hat{L}}$. Using the case $[L, L]$ simply connected we deduce that $\mathcal{F}_{\hat{Z}}$ has a reduction to $Z([\hat{L}, \hat{L}])$. Pushing to Z we obtain that \mathcal{F}_Z has a reduction to the image of $Z([\hat{L}, \hat{L}])$ in Z , namely to $Z([L, L])$. \square

Proof. [of Theorem 1.2] The determinant map

$$\det : \mathcal{M}_L^{\check{\lambda}_L} \rightarrow \mathcal{M}_{cZ}^{\det(\check{\lambda}_L)}$$

is clearly $\mathcal{M}_{Z(L)}^0$ -equivariant and both actions are transitive: for the LHS this is Corollary 3.1 and for RHS it comes from the fact that $Z(L)$ surjects onto cZ .

It remains then to check that the stabilisers are the same. Since $\ker(Z \rightarrow cZ)$ is precisely $Z([L, L])$ we have that the stabiliser on RHS is $\mathcal{M}_{Z([L, L])}$ so using Lemma 4.5 we conclude. \square

Remark 4.6. Given this simple description of $\mathcal{M}_L^{\check{\lambda}_L}$ one might be led to think that the existence of a universal bundle on it is automatic from the classical Poincaré bundle on the Picard variety. This is not the case. For example, if $Z(L)$ is not connected then Theorem 6.8 from [BH12] says that $\mathcal{M}_L^{\check{\lambda}_L}$ doesn't admit a universal bundle (called Poincaré bundle in loc.cit). On the other hand, the same theorem tells us that if $[L, L]$ is simply connected and $Z(L)$ is connected then there is a universal bundle. I haven't determined precisely what happens if $[L, L]$ is not simply connected, one of the issues being that the automorphism group of a stable L -bundle is bigger than $Z(L)$ in this situation.

5. APPENDIX

In this Appendix we provide a table (taken from [Fra16]) with the Levi subgroups L appearing in 1.1, as well as their relative Weyl groups $W_{L,G}$. We omit $\check{\lambda}_G = 0$ since in this case the Levi is always equal to the maximal torus.

G	deg	Type of M	Diagram of (G, M)	Type of $W_{M,G}$
A_{n-1}	d	$A_{n/e-1} \times \cdots \times A_{n/e-1}$ $e = \gcd(n, d)$	$\boxed{A_{n/e-1}} - \circ - \boxed{A_{n/e-1}} - \circ \cdots - \circ - \boxed{A_{n/e-1}}$	A_{e-1}
B_n	1	A_1	$\circ - \circ - \circ - \cdots - \circ - \Rightarrow \bullet$	C_{n-1}
C_{2n}	1	$\underbrace{A_1 \times A_1 \cdots \times A_1}_n$	$\bullet - \circ - \bullet - \circ \cdots \cdots \bullet \Leftarrow \circ$	C_n
C_{2n+1}	1	$\underbrace{A_1 \times A_1 \cdots \times A_1}_{n+1}$	$\bullet - \circ - \bullet - \circ \cdots \cdots \circ \Leftarrow \bullet$	C_n
D_{2n+1}	1	$A_1 \times \cdots \times A_1 \times A_3$	$\bullet - \circ - \bullet - \cdots - \circ - \bullet \begin{matrix} \swarrow \bullet \\ \searrow \bullet \end{matrix}$	C_{n-1}
	2	$A_1 \times A_1$	$\circ - \circ - \circ - \cdots - \circ - \circ \begin{matrix} \swarrow \bullet \\ \searrow \bullet \end{matrix}$	C_{n-1}
D_{2n}	(1,0)	$A_1 \times \cdots \times A_1$	$\bullet - \circ - \bullet - \cdots - \bullet - \circ \begin{matrix} \swarrow \bullet \\ \searrow \circ \end{matrix}$	B_n
	(0,1)	$A_1 \times A_1$	$\circ - \circ - \circ - \cdots - \circ - \circ \begin{matrix} \swarrow \bullet \\ \searrow \bullet \end{matrix}$	C_{2n-2}
	(1,1)	$A_1 \times \cdots \times A_1$	$\bullet - \circ - \bullet - \cdots - \bullet - \circ \begin{matrix} \swarrow \circ \\ \searrow \bullet \end{matrix}$	C_n
E_6	1	$A_2 \times A_2$	$\bullet - \bullet - \circ - \bullet - \bullet$ $\quad \quad \quad \downarrow \circ$	G_2
E_7	1	$A_1 \times A_1 \times A_1$	$\circ - \circ - \circ - \bullet - \circ - \bullet$ $\quad \quad \quad \downarrow \bullet$	F_4

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